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Algebraic Spinors and SUSY

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Using the Dirac–Kaehler formalism, we formulate the supersymmetric Wess–Zumino model. Special attention is paid to the proper definition of a two-dimensional spinor, its adjoint, and its generalization to other dimensions.

1. INTRODUCTION

The Dirac–Kaehler (DK) approach (Kaehler, 1962) to linearization of the Klein–Gordon equation has been considered by several authors (Plebañski, 1984; Becher and Joos, 1982). Two of the characteristics that make the DK an interesting alternative to the Dirac equation are the following: (i) It has a clear geometric interpretation (Becher and Joos, 1982). In particular, the spinor is seen as a coherent superposition of differential forms. (ii) It provides a natural framework to describe chiral fermions on the lattice (Becher, 1981).

The relation between differential forms and the DK spinors suggests the use of the latter in the formulation of supersymmetric theories (Banks *et al.*, 1982). In fact, this approach will open the interesting possibility of treating such theories in the lattice approximation. Indeed, the Wess–Zumino model in two dimensions has been constructed (Banks *et al.*, 1982; Elitzur and Schwimmer, 1983; Aratyn and Zimmerman, 1984). The extension of these considerations to four-dimensional models on the lattice depends on the understanding of the geometric properties of the fermions and the correct formulation of the corresponding Lagrangian in the continuum case.

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In this work we present a formulation of the SUSY Wess–Zumino model in four dimensions. In order to achieve this goal it was necessary to clarify the definition of a spinor. It is a well-known fact that the Dirac spinors are elements of a vector space on which the elements of the $SO(3,1)$ group, extended by parity, act irreductibly. In this approach to the Dirac spinors we know the way they transform under $SO(3,1)$; however, there is a nongeometrical characteristic associated to them.

Alternatively, some authors (Aratyn and Zimmerman, 1985; Plebañski, 1984) define the spinors in terms of the even part of a Clifford algebra and this has to be compared to the more conventional one in terms of a left ideal (Becher and Joos, 1982). As we shall see later, these definitions may coincide only in two dimensions.

The paper is organized in the following way: in Section 2 we make a brief introduction to algebraic spinors and work explicitly the case in which the vector space is taken to be the cotangent space-time. Section 3 contains a description of the DK formalism, the derivation of DK and adjoint DK equations, and its relation to the Dirac equation. In Section 4 we use the algebraic spinors in the formulation of the Wess–Zumino supersymmetric model.

2. ALGEBRAIC SPINORS

For an N-dimensional vector space V with nonsingular metric g we can choose a basis $\{\gamma^\mu, \mu = 1, 2, \ldots, N\}$ for which g is represented by the matrix $g^{\mu\nu}$. If the basis $\{\gamma^\mu\}$ is orthonormal, then $g^{\mu\nu}$ is diagonal of signature (p, q), with p and q the number of $+1$'s and -1's appearing in $g^{\mu\nu}$. We obtain a Clifford algebra C_N by introducing an associative and distributive product $\vee$ between the elements of this basis. This is a 2^N-dimensional algebra associated to V and g once the aforementioned product is restrained to satisfy the relation

$$\gamma^\mu \vee \gamma^\nu + \gamma^\nu \vee \gamma^\mu = 2g^{\mu\nu} \tag{1}$$

We denote the antisymmetric part of the Clifford product (conventionally called Grassman product) by $\wedge$, which, according to equation (1), is given by

$$\gamma^\mu \wedge \gamma^\nu \equiv \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]_\vee = \gamma^\mu \vee \gamma^\nu - g^{\mu\nu} \tag{2}$$

A basis for C_N can be expressed in terms of the Grassman product of γ^μ's:

$$\{1, \gamma^\mu, \gamma^{\mu_1\mu_2}, \ldots, \gamma^{\mu_1\mu_2\cdots\mu_N}\}$$

where

$$\gamma^{\mu_i\mu_k\cdots\mu_j} = \gamma^{\mu_i} \wedge \gamma^{\mu_k} \wedge \cdots \wedge \gamma^{\mu_j}$$

An algebraic spinor can be defined (Becher and Joos, 1982; Lounesto, 1986) as the left ideal of the Clifford algebra we have just constructed.

The connection between the algebraic and Dirac spinors is clarified if we observe that the elements of the algebra defined by

$$M^{\mu\nu} \equiv \tfrac{1}{4}[\gamma^{\mu}, \gamma^{\nu}]_{\vee}$$

satisfy the commutation rules

$$[M^{\rho\alpha}, M^{\alpha\beta}] = -g^{\rho\alpha}M^{\sigma\beta} + g^{\sigma\alpha}M^{\rho\beta} - g^{\sigma\beta}M^{\rho\alpha} + g^{\rho\beta}M^{\sigma\alpha}$$

which, for Minkowskian space-time, are the commutation rules of the $SO(3,1)$ group. Hence, we can conclude that algebraic spinors are a geometric generalization of Dirac spinors.

In order to give an illustrative example of a Clifford algebra and the construction of algebraic spinors, in this section we discuss the cotangent space-time and the associated Clifford algebra. Let dx^{μ}, $\mu = 1, \ldots, 4$, be a basis of the cotangent space of Minkowskian space-time. We introduce the Clifford product for every two elements of the basis as

$$dx^{\mu} \vee dx^{\nu} = g^{\mu\nu} + dx^{\mu} \wedge dx^{\nu} \tag{3}$$

where $\wedge$ denotes the usual exterior product for differential forms (Becher and Joos, 1982). Furthermore, we must also consider the Clifford product of an element of the basis dx^{μ} and a form Φ. This distributive and associative product is defined by

$$\begin{aligned} dx^{\mu} \vee \Phi &= dx^{\mu} \wedge \Phi + e^{\mu} \rfloor \Phi \\ \Phi \vee dx^{\mu} &= \Phi \wedge dx^{\mu} - e^{\mu} \rfloor \Phi \end{aligned} \tag{4}$$

where $e^{\mu}\rfloor$ stands for the contraction which is a linear operator defined by the following relations:

$$\begin{aligned} e^{\mu}\rfloor 1 = 0, \qquad e^{\mu}\rfloor dx^{\nu} &= g^{\mu\nu} \\ e^{\mu}\rfloor(\Phi \wedge \Psi) &= e^{\mu}\rfloor\Psi + (A\Phi)\rfloor\Psi \end{aligned} \tag{5}$$

where the main automorphism A is defined by

$$A\colon \quad \Lambda^{p} \mapsto \Lambda^{p}, \qquad A(\phi^{p}) = (-)^{p}\phi^{p}$$

Equations (3)–(5) complete the construction of the Clifford algebra for the cotangent space. To define an algebraic spinor, we will decompose this algebra in left and right ideals. With this aim we introduce the new basis Z_{ab} $(a, b = 1, \ldots, 4)$ given by [we use the notation of Becher and Joos (1982)]

$$Z = \sum_{H} \gamma_{H}^{T} B(dx^{H})$$

where γ^μ are Dirac matrices in an arbitrary representation and B is an antimorphism defined by

$$B(\phi^p) = (-)^{[p/2]}\phi^p$$

$[x]$ denotes the integer part of x. Every element of the Clifford algebra can be spanned in the following two equivalent ways:

$$\Phi = \sum_H \phi_H(x)\, dx^H; \qquad \Phi = \sum_{ab} \phi_{ab}(x) Z_{ab} \tag{6}$$

This basis has the properties

$$dx^\mu \vee Z_{ab} = \sum_c (\gamma^{\mu T})_{ac} Z_{cb} \tag{7}$$

and

$$Z_{ab} \vee Z_{cd} = 4 Z_{cb} \delta_{ad} \tag{8}$$

Using the relations (6) and (8), we arrive at the equation

$$\Phi \vee Z_{ab} = 4 \sum_c \phi_{ca} Z_{cb} \tag{9}$$

i.e., $\Phi \vee Z_{ab}$ belongs to the space generated by the basis Z_{cb} ($c = 1, \ldots, 4$) and fixed b and acts as a projector over this space. Thus a left ideal is a subspace generated by the elements of the basis Z_{ab} with fixed b. In a similar way, right ideals are the subspaces generated by the Z_{ab} basis with fixed a. For explicit calculations we will use matrices in the Weyl representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

For this representation of the matrices we arrive at the following expression for the Z basis:

$$\begin{aligned}
Z_{11} &= 1 + dx^{03} - idx^{12} - idx^{0123} \\
Z_{12} &= dx^{01} - idx^{02} + dx^{13} - idx^{23} \\
Z_{13} &= -dx^3 - dx^0 - idx^{012} + idx^{123} \\
Z_{14} &= -dx^1 + idx^2 + dx^{013} - idx^{023} \\
Z_{21} &= dx^{01} + idx^{02} - dx^{13} - idx^{23} \\
Z_{22} &= 1 - dx^{03} + idx^{12} - idx^{0123} \\
Z_{23} &= -dx^1 - idx^2 - dx^{013} - idx^{023} \\
Z_{24} &= dx^3 - dx^0 + idx^{012} + idx^{123}
\end{aligned} \tag{10}$$

$$
\begin{aligned}
Z_{31} &= dx^3 + dx^0 - idx^{012} - idx^{123} \\
Z_{32} &= dx^1 - idx^2 + dx^{013} - idx^{023} \\
Z_{33} &= 1 - dx^{03} - idx^{12} + idx^{0123} \\
Z_{34} &= -dx^{01} + idx^{02} + dx^{13} - idx^{23} \\
Z_{41} &= dx^1 + idx^2 - dx^{013} - idx^{023} \\
Z_{42} &= -dx^3 + dx^0 + idx^{012} - idx^{123} \\
Z_{43} &= -dx^{01} - idx^{02} - dx^{13} - idx^{23} \\
Z_{44} &= 1 + dx^{03} + idx^{12} + idx^{0123}
\end{aligned} \tag{10}
$$

The basis has the following interesting properties, which will be useful in the formulation of the four-dimensional Wess–Zumino model.

(i) The Clifford products, which in the dx^μ basis are rather involved, reduce in the Z_{ab} basis to matrix products

$$
\begin{aligned}
\Phi &= \sum_{ab} \phi_{ab}(x) Z_{ab}; \qquad \Psi = \sum_{cd} \psi_{cd}(x) Z_{cd} \\
\Phi \vee \Psi &= \sum_{abcd} \phi_{ab}\psi_{cd} Z_{ab} Z_{cd} = 4 \sum_{ad} (\phi\psi)_{ad} Z_{ad}
\end{aligned} \tag{11}
$$

(ii) Scalar content of the Clifford product. In the Z basis the projection over 0-forms (denoted $\langle , \rangle$) reduces to the trace of the coordinate matrix. Indeed, in the Z basis the only elements containing 0-forms are on the diagonal; therefore

$$
\langle \Phi \rangle = \left\langle \sum_{ab} \Phi_{ab} Z_{ab} \right\rangle = \sum_{ab} \phi_{ab} \delta_{ab} = \mathrm{tr}(\phi)
$$

Therefore for an arbitrary product $\langle \Phi \vee \Psi \rangle = 4\, \mathrm{tr}(\phi\psi)$.

Using the explicit construction of an algebraic spinor presented in this section, below we show the equivalence between the DK and four independent ordinary Dirac equations.

3. DIRAC–KAEHLER FORMALISM

The DK formalism arises as an alterntive to the Dirac approach to the linearization of the Klein–Gordon equation. In terms of the d and δ operators defined in the Grassman algebra of differential forms (Plebañski, 1984; Becher and Joos, 1982), the D'Alembert operator is written as

$$
\Delta = (d - \delta)^2
$$

Hence, the square root of the Klein–Gordon equation reads

$$(d - \delta + im)\Phi = 0 \tag{12}$$

where Φ is a linear combination of differential r-forms, i.e., it belongs to the Grassman algebra of differential forms. Equation (12) was proposed by Kaehler (1962) and it is known as the DK equation.

To write equation (12) in terms of the components with respect to the Z basis, we will use the relations

$$\begin{aligned} d\Phi &= dx^\mu \wedge \partial_\mu \Phi \\ \delta\Phi &= -e^\mu \rfloor \partial_\mu \Phi \end{aligned} \tag{13}$$

Then the DK equation becomes

$$\begin{aligned} (d - \delta + im)\Phi &= (dx^\mu \vee \partial_\mu + im) \sum_{ab} \phi_{ab} Z_{ab} \\ &= \sum_{abc} \partial_\mu \phi_{ab} (\gamma^\mu)_{ca} Z_{cb} + im \sum_{cb} \phi_{cb} Z_{cb} \\ &= \sum_{cb} [(\gamma^\mu \partial_\mu + im)\phi]_{cb} Z_{cb} \\ &= 0 \\ &\Rightarrow (\gamma^\mu \partial_\mu + im)\phi = 0 \end{aligned} \tag{14}$$

From this equation we conclude that elements of left ideals satisfying the DK equation are spinors which satisfy ordinary Dirac equations.

In a similar way we can show that the solutions of the adjoint Dirac equation

$$\partial_\mu \Phi \vee dx^\mu - im\phi = 0 \tag{15}$$

are elements of right ideals of the Clifford algebra. In fact, there exists a simple relation between the solutions of the DK equation and the solutions of the adjoint DK equation in terms of the main morphisms, which to our knowledge has not been previously reported in the literature. Indeed, using equation (4) and $dx^\mu \wedge A\Phi = \Phi \wedge dx^\mu$, we obtain

$$\partial_\mu \Phi \vee dx^\mu = dx^\mu \wedge \partial_\mu (A\Phi) - e^\mu \rfloor \partial_\mu (A\Phi) = (d + \delta)A\Phi$$

Therefore equation (15) becomes

$$((d + \delta)A - im)\Phi = 0$$

Applying the main morphisms to this equation and using the relations

$$Ad = -dA, \qquad A\delta = -\delta A, \qquad Bd = dAB, \qquad B\delta = -\delta AB$$

we finally obtain the adjoint DK equation

$$(d - \delta + im)AB\Phi = 0$$

Clearly, the action of AB on Φ generalizes the well-known definition of the adjoint Dirac spinor

$$\bar{\Phi} = AB\Phi$$

We are now in a position to construct a basis-independent Lagrangian describing half-spin particles in terms of a superposition of r-forms. Furthermore, we know that differential forms, or their components, are the natural mathematical tools to describe boson fields. Hence, the formulation of a supersymmetric model in terms of the elements of the Clifford algebra arises naturally.

4. DIRAC–KAEHLER APPROACH TO THE WESS–ZUMINO MODEL

4.1. Two-Dimensional Model

Aratyn and Zimmerman (1984) (AZ) used the DK approach to formulate the two-dimensional Wess–Zumino model. They assumed that fermions are described by the even elements of a differential form

$$\Psi = f_0 + \frac{1}{2!} f_{\mu\nu} dx^\mu \wedge dx^\nu = f_0 + f_{12} dx^{12}$$

$$\Psi^* = f_0^* + f_{12}^* \, dx^{12}$$

whereas Φ is connected to the bosonic field

$$\Phi = \phi_\mu dx^\mu; \qquad \Phi^* = \phi_\mu^* dx^\mu; \qquad \phi_\mu = \partial_\mu \phi$$

In this approach the Wess–Zumino Lagrangian density is given by (Aratyn and Zimmerman, 1984)

$$\begin{aligned} L = P_0[&\Phi^* \wedge \Phi + 2i\Psi^*(d - \delta)dx^2 \vee \Psi - W'^2 \\ &- i\Psi^* \vee dx^2 \vee \Psi \vee (dx^{12} + 1) \vee dx^2 W'' \\ &- i\Psi^* \vee dx^2 \vee \Psi(dx^{12} - 1) \vee dx^2 W''^*] \end{aligned}$$

where $W(\phi)$ is the superpotential, which is an analytic function of the scalar field ϕ, P_0 denotes the projection on zero forms, and $W' = dW/d\phi$, etc.

Correspondence with the usual two-dimensional Wess–Zumino model can be achieved by identifying $\chi_1 = f_0 + f_{12}$, $\chi_2 = f_0 - f_{12}$ as the components of a two-dimensional spinor. Aratyn and Zimmerman correctly pointed out that the components of this spinor have different flavor, which in the language we are using means that the components of the spinor belong to different left ideals. However, as we have shown in the previous section, a Dirac field must be described by a left ideal.

An intriguing possibility is the extension of the DK approach to the Wess–Zumino model to other dimensions. It turns out, however, that the correspondence between the description of fermionic fields as a combination of the even elements of a differential form and the one in terms of left ideals of a Clifford algebra is valid only in two dimensions.

Indeed, in two dimensions we can choose the matrices as

$$\gamma_1 = i\sigma_1; \quad \gamma_2 = i\sigma_2$$

where σ_i are the Pauli matrices and the metric is given by

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore the Z matrix is

$$z = \begin{pmatrix} 1 - dx^{12} & i(dx^1 - dx^2) \\ -(dx^1 + dx^2) & 1 + dx^{12} \end{pmatrix}$$

which for a general

$$\Psi = f_0 + f_\mu dx^\mu + \frac{1}{2!} f_{\mu\nu} dx^\mu \wedge dx^\nu$$

implies

$$\psi = \begin{pmatrix} f_0 - f_{12} & i(f_1 - f_2) \\ i(f_1 + f_2) & f_0 + f_{12} \end{pmatrix}$$

Notice that the components of the AZ spinor belong to different left ideals. It is easily seen that the correspondence between the Wess–Zumino model and the AZ formulation is due to the following facts: (a) Z contains even elements only on the diagonal. Therefore $\psi^* = \psi$. (b) $\gamma^2\gamma^\mu$ is diagonal. These characteristics are valid only in two dimensions. Therefore in higher dimensions a new formulation is necessary. Before going to four dimensions, let us remark that in two dimensions we deal with two ideals and also with two even elements and thus a one-to-one correspondence can be established among them. In more than two dimensions such a one-to-one correspondence is not possible. Thus, for example, in four dimensions we have eight even elements and only four ideals. Moreover, once the elements of different left ideals are combined, the proper transformations under the Lorentz group are not ensured.

4.2. Four-Dimensional Wess–Zumino Model

In terms of equation (16) and the projection over zero forms discussed below equation (11), the four-dimensional supersymmetric Wess–Zumino model is given by

$$L = L_K + L_H + L_Y$$

with

$$L_K = \frac{1}{2}\partial_\mu \Phi \vee \partial^\mu \Phi^* + \frac{i}{2}\langle \bar{\psi} \vee (d-\delta)\psi \rangle$$

$$L_H = W' \vee F_- + W'^* \vee F_+ + \frac{1}{2} F_+ \vee F_-$$

$$L_Y = -W''\langle \bar{\Psi} \vee \Psi_- \rangle - W''^*\langle \bar{\Psi} \vee \Psi_+ \rangle$$

where

$$\bar{\Psi} = AB\Psi; \qquad \Psi_\pm = (1 + idx^{0123})$$

with $F_\pm$ the usual auxiliary fields.

Using the Z basis, it is straightforward to convince oneself that this expression reduces to the Wess–Zumino Lagrangian.

For completeness we list below the SUSY transformations:

$$\delta\Phi = 2\langle \bar{\epsilon} \vee \Psi_- \rangle \qquad \delta\Psi^* = 2\langle \bar{\epsilon} \vee \Psi_+ \rangle$$

$$\delta F_- = -2i\langle \bar{\epsilon} \vee (d-\delta)\Psi_- \rangle \qquad \delta F_+ = -2i\langle \bar{\epsilon} \vee (d-\delta)\Psi_+ \rangle$$

$$\delta\Psi_- = \bar{\epsilon} \vee F_- - id\Phi \vee \epsilon_+ \qquad \delta\bar{\Psi}_- = \bar{\epsilon}_- \vee F_+ + i\bar{\epsilon}_+ \vee d\Phi^*$$

$$\delta\Psi_+ = \epsilon_+ \vee F_+ - id\Phi^* \vee \epsilon_- \qquad \delta\bar{\Psi}_+ = \bar{\epsilon}_+ \vee F_- + i\bar{\epsilon}_- \vee (d-\delta)\Phi$$

5. CONCLUSIONS

We have discussed the algebraic spinors and their use in the formulation of four-dimensional SUSY models. [Extensions to other dimensions are straightforward because the main characteristics of the Z basis are preserved and the key relation, equation (8), remains unchanged except for some multiplicative factors.] In particular, we have clarified the difference with the (inconsistent) description in terms of even elements of the Clifford algebra. We also introduced the $\bar{\Phi}$ [equation (16)] which fulfills the adjoint DK equation and greatly simplifies the formulation in arbitrary dimensions.

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